CS4860 Preliminary Exam Tuesday October 16, 2012

This is a closed-book no-notes exam covering the first half of the course. Please write neatly and carefully in the exam booklet. Put your name on each booklet.

Question 0. Propose at least one topic for your course project. Say why it is related to the course material and what you plan to accomplish. Note, we do not expect that these projects need to be more than a couple days of work. They can be more if you are aiming for a very high grade or are especially keen on a topic. The project should be written up in a professional way.

Question 1. On Implication (5 parts)

In the Boolean truth value explanation of <u>implication</u> there seems to be very minimal "connection" between the antecedent P and the conclusion Q in $P \supset Q$. In the evidence semantics we use a function, say $\lambda x. q(x)$, that relates assumed evidence x for P to "computed" evidence $q(x)$ for Q. So if we are given evidence p for P, then $q(p)$ evaluates to evidence for Q. Thus the explanation that $\lambda x.x$ is evidence for $P \supset P$ is very clear. Using this constructive meaning of $P \supset Q$, explain the truth functional meaning by answering these questions and then in the last part giving your best explanation in words, e.g. in part (e).

(a) Write the truth table for $P \supset Q$ and explain each case where $P \supset Q$ gets the value t in terms of evidence.

(b) Explain the case where $P \supset Q$ is assigned false, f.

(c) The the Boolean semantics of $P \supset Q$ is the same as for $\sim P \vee Q$. Explain why this is not equivalent to the evidence interpretation by <u>comparing the two evidence</u> meanings – the one for $P \supset Q$ and the one for $\sim P \vee Q$.

(d) How are $P \supset Q$ and $\sim P \vee Q$ related in evidence semantics? That is, which one implies the other? Illustrate your answer in the case where $P = (0 = 1)$ and Q = Goldbach's Conjecture.

(e) Explain how we make sense of the truth table interpretation in terms of evidence.

Question 2 Extended truth tables (3 parts)

Logicians tried to explain constructive logic in terms of truth tables by adding a third "truth value" called unknown. Let's use t, f, u for these three values.

(a) What would be sensible values in the following "truth table"?

P	Q	P ∧ Q	P ∨ Q	P ⊃ Q	~ P
t	t	t	t	t	f
t	f	f	t	f	f
t	u	1	6	11	f
f	t	f	t	t	t
f	f	f	f	t	t
f	u	2	7	12	t
u	t	3	8	13	16
u	f	4	9	14	17
u	u	5	10	15	18

First group any cases that are clearly the same, e.g. 1 and 3, 2 and 4, 16, 17, 18, then 6, 8, etc.

(b) Does this truth table supply t as the value in all cases where the evidence semantics would for these valid formulas, e.g. using the P ∨ ~P and Q ∨ ~Q assumptions?

1. ~ (P & ~P)
2. P ⊃ P
3. P ⊃ (Q ⊃ P)

(c) Why is this semantics less satisfactory for P ⊃ Q than evidence semantics?

Question 3 Counter models and Absence of Evidence

We have one very simple way to explain why some formulas have no evidence terms (also called realizers).

(a) Illustrate this method on the fallacies listed below

1. $(P \supset Q) \supset (\sim P \supset \sim Q)$

2. $(P \vee Q) \supset P$

3. $\sim (P \wedge Q) \supset \sim P$

(b) Why is it more difficult to show that $(\neg\neg P \supset P)$ and $((P \supset Q) \supset P) \supset P$ are not realizable, e.g. have no evidence?

(c) How do we know that $(P \wedge \sim P)$ is not provable by Refinement Logic?

Question 4 König's Lemma (4 parts)

In Ch. III §1 p. 32 Smullyan states and proves König's Lemma from 1936. Here is his statement as proved in lecture.

Every finitely generated tree $\mathcal{T}$ with infinitely many points must contain at least one infinite branch.

(1) Prove this theorem for binary trees. (Note by Smullyan p.3 we can consider these to be ordered trees with the left branch as 1st, the right as 2nd.)

(2) Explain (informally) why this result is not constructive.

(3.) State the contrapositive of this theorem, called the Fan Theorem. (Note, we claimed that this result is constructive if the paths are "computably generated" as we will explain later in the course.)

Recall, if $P \supset Q$, the contrapositive is $\sim Q \supset \sim P$.

(4) Provide the evidence term for $(P \supset Q) \supset (\sim Q \supset \sim P)$.

Question 5 Syllogisms (3 parts)

One of the oldest and most cited logical arguments is Aristotle's syllogism. On the left is the original, on the right a more modern phrasing.

1. All men are mortal	1. All humans are mortal
2. Socrates is a man	2. Socrates is a human
3. Therefore Socrates is mortal	3. Socrates is mortal.

Let's abbreviate Human(x) by H(x) and Mortal(x) by M(x). Line 1 can be expressed in First-Order Logic (FOL) as

$$\forall x.(H(x) \supset M(x))$$

The first-order models of the world typically have objects in them that we name with nouns, some can be proper nouns such as Socrates, Plato, Athens, etc. they have properties given by predicates such as Human(x), City(x), etc. We call the part of the world that a sentence is talking about the <u>domain of discourse</u> D or <u>universe of individuals</u> (Smullyan p. 46).

Let's say that domain D_0 has individuals $s, p, a, \ldots$ for Socrates, Plato, Aristotle, etc. So $D_0 = \{s, p, a, \ldots\}$. There can be other objects as well such as cities, tables, chairs, robots, etc. (Let's have a robot r just for fun.)

We will claim $H(s)$, $H(p)$, $H(a)$, and $H(r)$ as atomic predicates with explicit evidence, say $hs \in [H(s)]$, $hp \in [H(p)]$. (For fun, $rr \in \boxed{Robot(r)}]$ and $H(r)$ is empty.)

(1) State the above syllogism as a single formula of FOL with the terms $s, p, a, \ldots$

(2) Give evidence for this syllogism in domain D_0. continued

(3) Use truth functional semantics from Smullyan and lectures to <u>refute</u> the fallacious claim: "Socrates is mortal, therefore he is a man" based on premise 1. To do this you can build a "fictional domain of discourse" for the purpose of logic. Building interesting models of first-order formulas is part of the fun of the subject of <u>model theory</u>. We will see later certain "non standard models" of arithmetic. In the case of this problem, you can think of a non standard notion of mortal objects.

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Question 6 (extra credit, also problem for HW4) (3 parts)

Here is a proposition that is truth functionally valid, the question is what is the lowest dimension for the constructive proof?

$$* \left(\exists z.\, I(z) \wedge (\forall x.(I(x) \wedge Q(x)) \supset \exists y.\, R(y)) \wedge (\forall x.(I(x) \wedge \sim Q(x)) \supset \exists y.\, R(y))\right)$$
$$\supset \exists y.\, R(y)$$

1. Explain why an instance (at least one perhaps more) of $P \vee \sim P$ is needed to provide evidence for $*$.

2. Add the necessary evidence among the hypotheses by adding to one or more of the three hypotheses. Do this as economically as possible, e.g. make the smallest addition that is sufficient.

3. Provide the evidence term for the augmented formula and discuss the dimension of this evidence based on our definition of dimension for IPC formulas.